Optical beam and its operation in low dimensional space

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Abstract: This paper proposes the concept of low dimensional optical beam and operator. In low dimensional space, beam (or operator) is decomposed into a limited number of orthogonalized low dimensional beams (or operators) through the singular value decomposition. It is possible to generate an unconventional beam by these low dimensional beams. Low dimensional operator allows independent operation of orthogonal dimensions which may produce greater freedoms. Storage space and computation resource are saved dramatically by using this method. Experimental realization of this scheme is briefly discussed at the end.

OCIS codes: (000.3860) Mathematical methods in physics; (140.3430) Laser theory; (190.4410) Nonlinear optics, parametric processes.

References and links

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1. Introduction

Recently there is an increasing interest in unconventional optical beams due to their special properties, such as Airy beam, dark-hollow beam and vortex beam. Generation and operation of these unconventional beams are not a simple work, but there's always the possibility that these beams can be decomposed into more simple beams [1, 2]. Simple beam is easier to be generated and operated, and it reflects substantive characteristics of the beam. Unconventional beam generators are reported by use of quasi-phase-matching nonlinear crystals [3, 4] and surface plasmonic structures [5]. Both are one dimensional generators and generate one dimensional beams. It is possible to compose these low dimensional operators and low dimensional beams to generate real two dimensional(2D)unconventional beams. In this paper, we propose a method to decompose 2D beams to low dimensional beams by using the singular value decomposition (SVD). Firstly, we describe this method mathematically and decompose a Laguerre-Gaussian beam to illustrate the process and its properties. A vortex Bessel-Gaussian beam is decomposed into simple low dimensional beams, suggesting a novel method that generates high dimensional beams by composition of low dimensional beams. And this method can save the computation resource and the storage space. Operators of beams can also be decomposed, for example, lensing and free space propagation. By using this method we calculate the focus of a slab beam with independent zoom of x and y axis, which can not be realized by using traditional fast Fourier transform (FFT) method. Finally, we discuss the experimental realization of decomposition and composition of beams.

2. Theory

In linear algebra, the singular value decomposition (SVD) of an $m \times n$ matrix $\mathbf{M}_{m\times n}$ is a factorization of the form $\mathbf{M}_{m\times n} = \mathbf{U}_{m\times m}\mathbf{S}_{m\times n}\mathbf{V}^*_{n\times n}$, where $\mathbf{U}$ is an $m \times m$ complex unitary matrix, $\mathbf{S}$ is an $m \times n$ rectangular diagonal matrix with non-negative real numbers on the diagonal, and $\mathbf{V}^*$(the conjugate transpose of $\mathbf{V}$) is an $n \times n$ complex unitary matrix. The diagonal entries $S_{k,k}$ or $\mathbf{S}$ are known as the singular value of $\mathbf{M}$. A common convention is to list the singular values in descending order. In this case, the diagonal matrix $\mathbf{S}$ is uniquely determined by $\mathbf{M}_{m\times n}$. The m columns of $\mathbf{U}$ and the n columns of $\mathbf{V}$ are called the left-singular vectors and right-singular vectors of $\mathbf{M}$, respectively.

Here we define $\vec{U}_{:,k}$ and $\vec{V}_{:,k}$ as k-th column vector of $\mathbf{U}_{m\times m}$ and $\mathbf{V}_{n\times n}$, respectively. The SVD is written as:

$$\mathbf{M}_{m\times n} = \mathbf{U}_{m\times m}\mathbf{S}_{m\times n}\mathbf{V}^*_{n\times n} = \sum_{k=1}^{r} \vec{U}_{:,k} S_{k,k} \vec{V}^*_{:,k} = \sum_{k=1}^{r} \check{\mathbf{M}}_k. \tag{1}$$

The rank of matrix $\check{\mathbf{M}}_k = \vec{U}_{:,k} S_{k,k} \vec{V}^*_{:,k}$ is 1, and we call this matrix the k-th rank-one matrix (ROM) of $\mathbf{M}_{m\times n}$. The physical significance of SVD is clear: the singular values represent the proportion of ROMs in the original matrix ($\mathbf{U}$ and $\mathbf{V}$ are unitary matrices).

Traditionally, only the electric field of optical beam is dealt with since the magnetic field component is essentially the same. Under rectangular coordinate system, without considering the state of polarization, a 2D beam is expressed as $E(x,y)$ (propagating along z-axis). To deal with polarization information, vectors with any polarization can be decomposed to the scalar version expressed as $Ex(x,y)$ and $Ey(x,y)$, which represent the x and y direction polarization component, respectively. Two dimensional matrix $\mathbf{E}_{m\times n}$ is the discretization of $E(x,y)$. If $\mathbf{E}_{m\times n} = \check{\mathbf{E}}_k$ is a rank-one matrix, we have $\check{\mathbf{E}}_k = S_{k,k}\vec{U}_{:,k}\vec{V}^*_{:,k}$, where $\vec{U}_{:,k}$ is the discretization of $E(y)$, and $\vec{V}^*_{:,k}$ is the discretization of $E(x)$. The corresponding $\check{E}(x,y) = S \cdot E(x)E(y)$ is called rank-one beam (ROB), and S is singular value of $\check{E}(x,y)$. If $E(x,y)$ is not a ROB, we have

$$E(x,y) = \sum_{k=1}^{r} \check{E}_k(x,y) = \sum_{k=1}^{r} S_k \cdot E_k(x)E_k(y), \tag{2}$$

where $\check{E}_k(x,y)$ is the k-th ROB, and S_k is its singular value.

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Operation of optical beam, such as Fourier transformation, lensing and free space propagation, can also be decomposed according to the above procedures. Suppose $\mathscr{H}$ is a linear operator, that is:

$$\mathscr{H}(A+B) = \mathscr{H}(A) + \mathscr{H}(B). \tag{3}$$

If s is a complex number,

$$\mathscr{H}(sA) = s\mathscr{H}(A). \tag{4}$$

If the following equation is tenable:

$$\check{\mathscr{H}}(A(x)B(y)) = \check{\mathscr{H}}(A(x))\check{\mathscr{H}}(B(y)), \tag{5}$$

$\check{\mathscr{H}}$ is called a rank-one operator (ROO). For example, Fourier transform and lensing are rank one operators. If the rank of an operator $\mathscr{H}$ is more than one, it can be factorized as:

$$\mathscr{H} = \sum_{q=1}^{s} \check{\mathscr{H}}_q. \tag{6}$$

Finally, the full expression of the beam's operation in low dimensional space is written as:

$$\mathscr{H}\mathbf{E}(x,y) = \sum_{k=1}^{k=r}\sum_{q=1}^{q=s} \check{\mathscr{H}}_q \check{\mathbf{E}}_k = \sum_{k=1}^{k=r}\sum_{q=1}^{q=s} S_k \check{\mathscr{H}}_q E_k(x) \check{\mathscr{H}}_q E_k(y). \tag{7}$$

Here we call $\check{\mathscr{H}}_q E_k(x)$ or $\check{\mathscr{H}}_q E_k(y)$ lower dimensional operation of optical beam $\mathbf{E}(x,y)$. For free space propagating, a two dimensional optical beam $\mathbf{E}(x,y)$ propagating in three space X-Y-Z will be decomposed to one dimensional optical beams $E_k(x)$ and $E_k(y)$, which propagate in two dimensional space X-Z and Y-Z, respectively, as shown in Fig. 1.

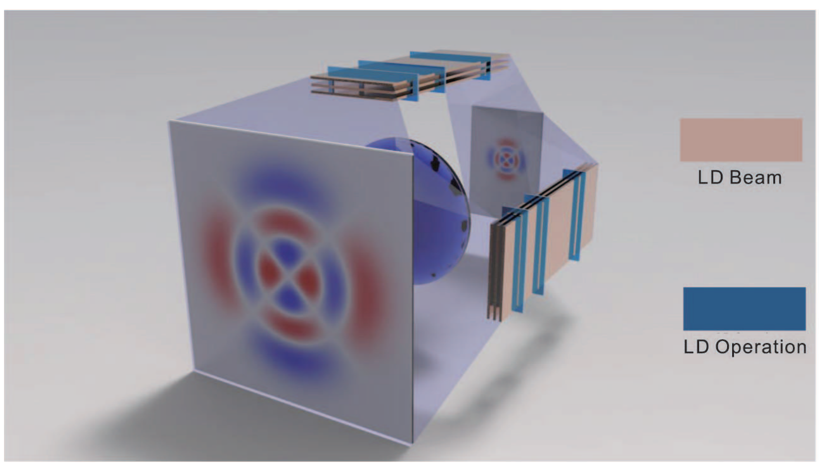

Fig. 1. Low dimensional (LD) lensing and propagation of optical beam.

3. Decomposition of optical beams

The electric beam of Hermite-Gaussian mode is essentially given by the product of a Gaussian function and a Hermite polynomial as shown in Eq. (8). From the expression it is obviously a rank one beam regardless of the value of index m and n.

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$$\mathbf{E}_{mn}(x,y) = E_0 H_m(\frac{\sqrt(2)x}{\omega})H_n(\frac{\sqrt(2)y}{\omega})exp(-\frac{x^2+y^2}{\omega}). \tag{8}$$

Laguerre-Gaussian mode optical beam is expressed as:

$$\mathbf{E}_{ps}(x,y) = E_0 \rho^s L_p^s(\rho^2)exp(-\rho^2/2)cos(s\theta), \tag{9}$$

where $L_p^s(\rho^2)$ is generalized Laguerre polynomials, $\rho = \sqrt{2}r/\omega = \sqrt{2(x^2+y^2)}/\omega$, and θ is azimuth angle. Figure 2(a) shows the Laguerre-Gaussian beam $\mathbf{E}(x,y)$ with $p = 2$ and $s = 2$.

For a given beam $E(x,y)$, discretization should be taken as:

$$E_{jk} = E(x = \frac{jL_x}{m}, y = \frac{kL_y}{n}), \tag{10}$$

where $j = 1,2,...,m$, $k = 1,2,...,n$ and L_x, L_y is sample window width of x-direction and y-direction, respectively. Of course the sample window can be easily moved to (c_x, c_y) by coordinate transformation $(x = x - c_x, y = y - c_y)$. For any matrix $\hat{M}$, the sum of squares of the singular values equals the Frobenius norm, that is

$$\sum_{k=1}^{r} S_k^2 = \sqrt{\sum_{j,k} M_{jk}^2} = ||\mathbf{M}||_F. \tag{11}$$

Singular values will increase with sample number $m \times n$:

$$\mathbf{S} \propto \sqrt{m \times n}. \tag{12}$$

From the following calculation, we will find that normalized singular values and ROBs are intrinsic properties of the given beam, and will not change fundamentally with discretization process.

We decompose Fig. 2(a) to ROBs, and the first 4 ROBs are shown from Fig. 2(c) to Fig. 2(f). Figure 3 is the corresponding normalized singular values. Regardless of the dimensions of the discrete matrices (64×64, 128×128 and 256×256), the rank keeps 4 (the singular values of other ranks are much smaller than the first 4 ones, and it is caused by calculation noise), and the normalized singular values of the same order are equal. It should be pointed out that Laguerre-Gaussian beam with $p = 2$ and $s = 2$ can be decomposed to 4 Hermite-Gaussian modes which are ROBs. The singular value of 128×128 matrix is double that of 64×64 matrix, and so is the relationship between 256×256 and 128×128 matrices, which is consistent with Eq. 12. The singular value of the first 2 ROBs is nearly 5 times as large as the next two ones, which means that the composed beam by the first 2 ROBs will be similar to the original one, as shown in Fig. 2(b). The beam composed by the first 4 ROBs is just the same as Fig. 2(a), and the maximum difference between composed one and the original one is 10^{-13}, which is shown in Fig. 4.

Bessel-Gauss beam is non-diffractive. Recently, research on the propagation properties of Bessel vortices has been reported [6]. The vortex Bessel Gauss-beam can be expressed as [7]

$$E(\rho) = J_0(\alpha\rho)\exp[-(\frac{\rho}{\omega_0})^2]\exp(-i\theta), \tag{13}$$

where $\rho = \sqrt{x^2+y^2}$, the parameter $\alpha = k\sin\theta$ is the radial spatial frequency defined as the angle made by the conic wavefront to the z axis, k is wave vector, J_0 is zero order Bessel Function, ω_0 is beam waist and θ is the azimuth angle. We set the calculation window of 5×5 mm, parameter $\omega_0 = 0.8mm$ and $\alpha = 10^4$. Beam amplitude (complex modulus, maximum value is normalize to 1) is shown in Fig. 5(a) and phase in Fig. 5(b), respectively. Firstly, the beam is

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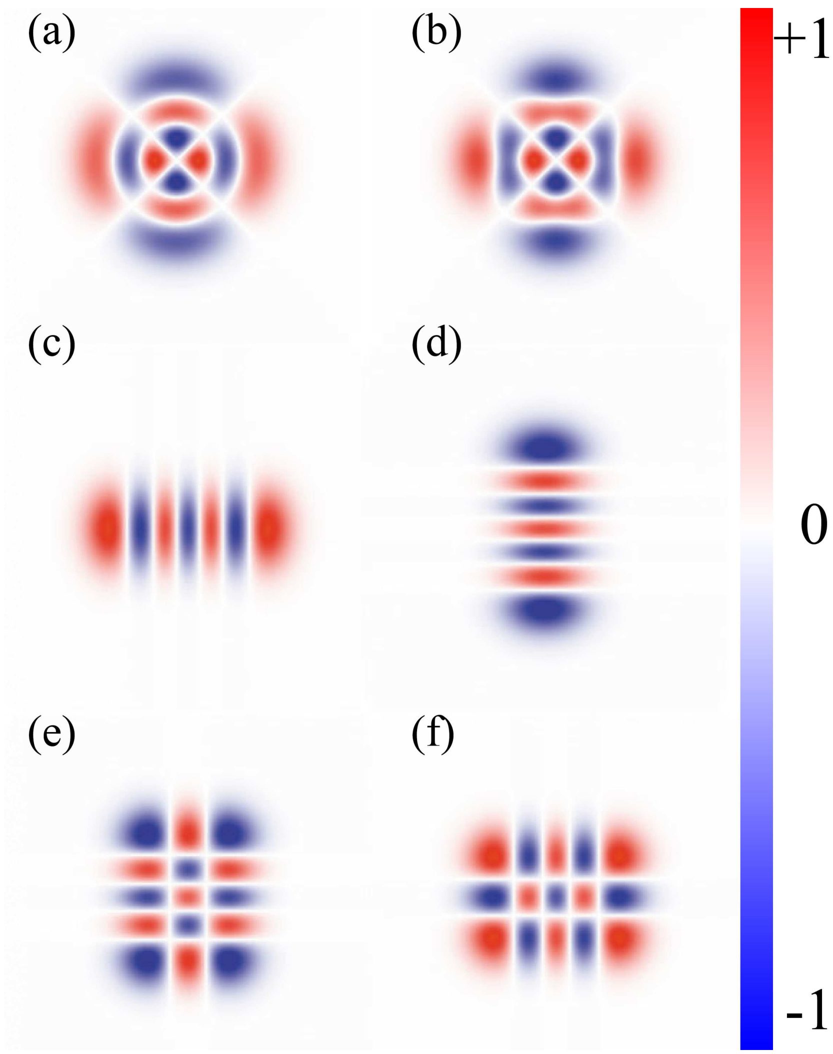

Fig. 2. (a) Original Laguerre-Guassian mode with $p = 2$ and $s = 2$. (b) Composition of the first 2 ROBs. (c)-(f) First 4 ROBs.

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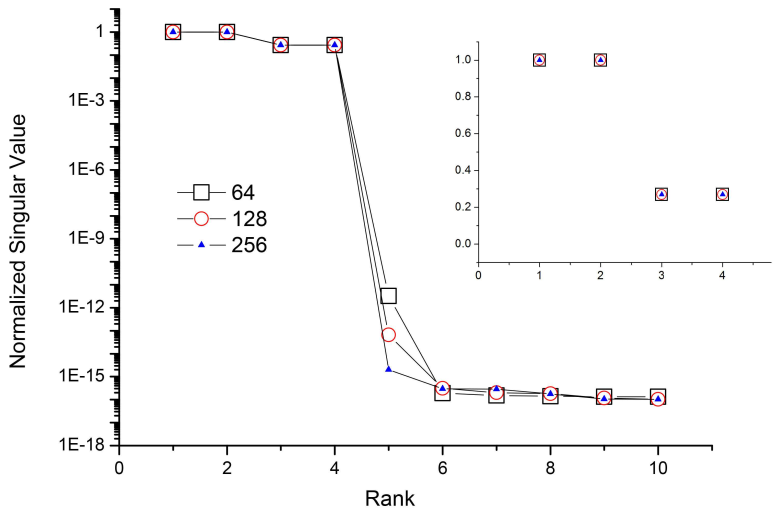

Fig. 3. Singular values of beam shown in Fig. 2(a).

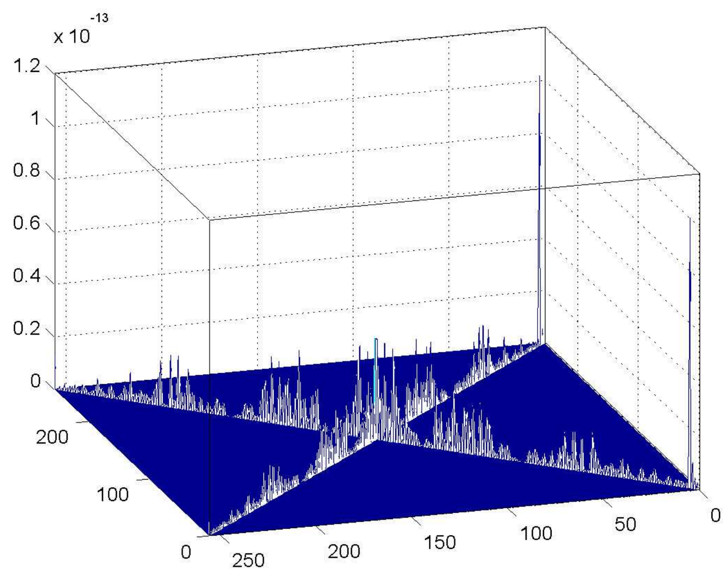

Fig. 4. Difference between the original beam and the composed beam.

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discretized to 128×128, 256×256 and 512×512 matrices and the singular value is calculated by means of SVD. As is shown in Fig. 6, it is obvious that the main singular values of the beam are irrelevant with the dimensions of matrices. The composed modulus and phase by the first 25 ROBs of the beam are shown in Fig. 5(c) and 5(d), respectively. Both of them are similar with the original beam. The phase deformation mainly exists in those fields where the modulus is small and those fields have slight effects on the essential properties of the beam. At the same time, it is noted that the composed beam by the first 50 ROBs has little difference with the original one. For the 512×512 discretized beam, if we take the first 25 low dimensional beam for storage, only $2 \times 25 \times 512$ storage space is needed, which is one tenth of the previous storage space. In data processing, for example, 2D Fourier transformation, the computation is the same with 2×512 1D Fourier transformation. However, by the method of SVD, only 2×25 1D Fourier transformation is needed and is one twentieth of the previous. Obviously, both computation and storage space are decreased.

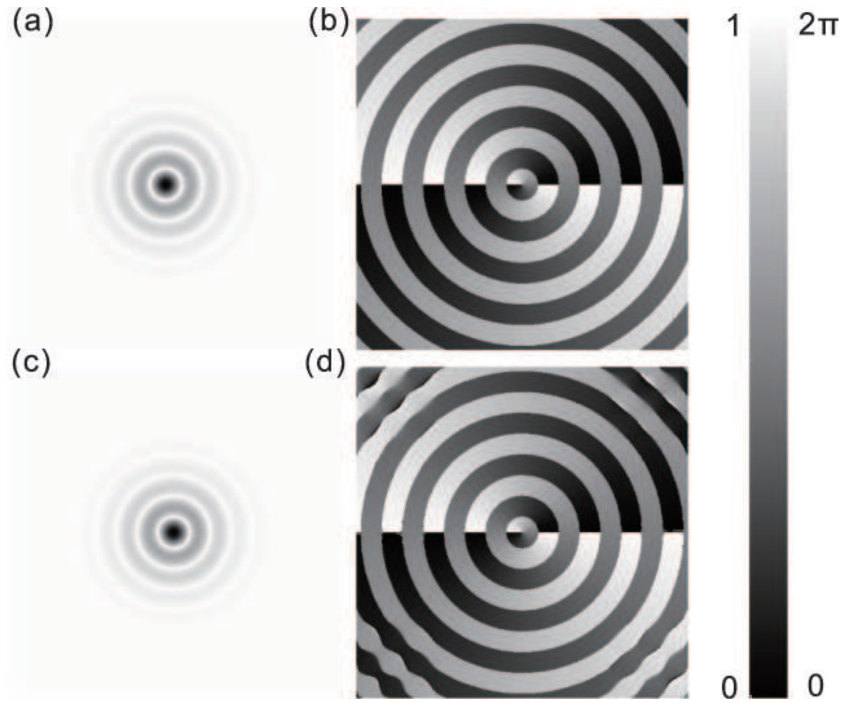

Fig. 5. (a) and (b) Modulus and phase of Bessel-Gaussian beam. (c) and (d) Modulus and phase of composed beam.

4. Decomposition of operators

Operations including Fourier transform, lensing and free space propagation can also be decomposed to ROOs. Fourier transform is an important operation of optical fields which is widely used in Fourier optics, image processing techniques and optical information processing, etc. A two-dimensional Fourier transform is written as:

$$F(f_x, f_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) exp(-i2\pi f_x x - i2\pi f_y y) dx dy. \tag{14}$$

Obviously, two dimensional Fourier transform is a ROO. Lensing operation is written as:

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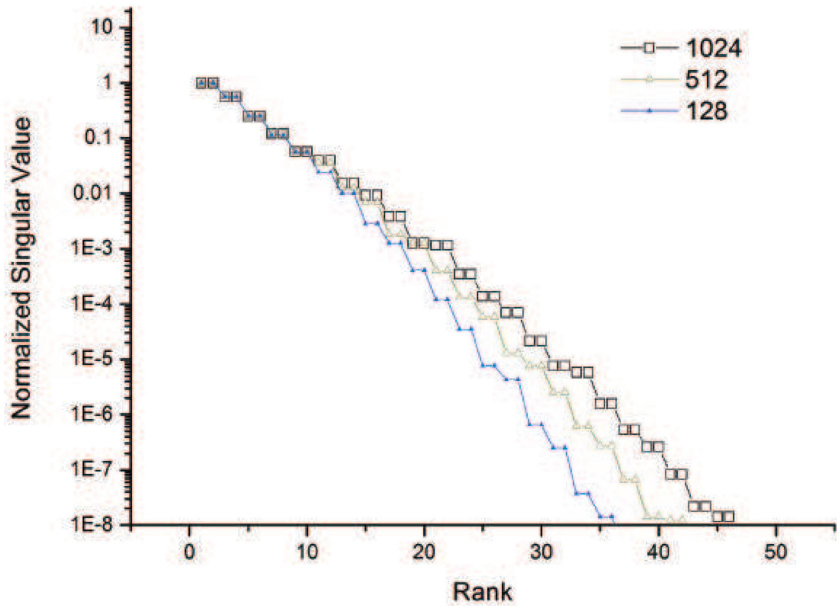

Fig. 6. Singular values of vortex Bessel-Gauss beam.

$$F(x,y) = f(x,y)exp[-k(\frac{x^2}{2f_x} + \frac{y^2}{2f_y})], \tag{15}$$

where f_x and f_y are the focus length of x-axis and y-axis, respectively, and $k = 2\pi/\lambda$ is the wave vector. Lensing is also a ROO.

According to angular spectrum theory, in frequency space the propagating operator can be expressed as

$$F(x,y) = f(f_x, f_y)exp[\frac{2\pi z}{\lambda}\sqrt{1-\lambda^2(f_x^2+f_y^2)}], \tag{16}$$

where f_x and f_y are spatial frequency, z is propagation length and λ is wavelength. The phase of this operator is shown in Fig. 7. It is not a ROO since x and y cannot be seperated independently. Figure 8 shows the decomposed singular values of K-operator. It has no more than 5 ROOs, and the major component concentrates at the first ROO.

5. Focusing of a strip shaped beam

In focusing calculation, if the FFT process is straightforwardly applied, it requires that the input and output beams be sampled on the same equally spaced transverse grid. This means that in tight focusing, the focused spot may be too small to cover enough sample points. Sziklas et al. proposed coordinate transformation to solve this problem under Fresnel approximation [8]. In his theory, the following coordinate transformation is taken:

$$\begin{aligned} x' &= \frac{\alpha}{z}x \\ y' &= \frac{\alpha}{z}y \\ z' &= \frac{\alpha^2}{zz_0}(z-z_0), \end{aligned} \tag{17}$$

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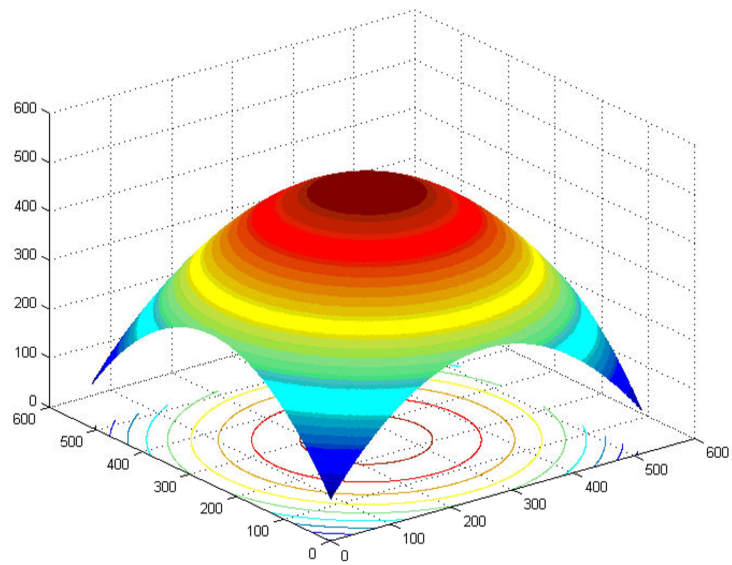

Fig. 7. Phase of the Kirchhoff operator.

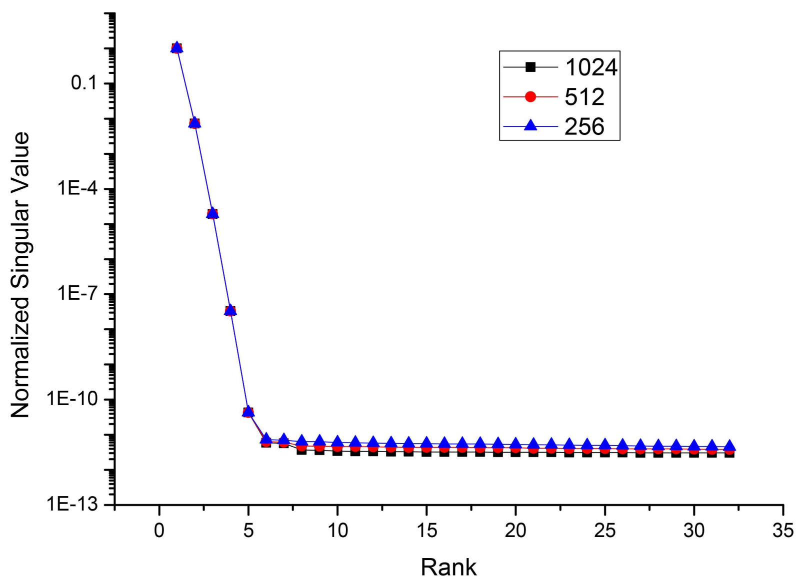

Fig. 8. Singular values of Kirchhoff operator.

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where $z_0 = R_0$ is the wavefront curvature, and α is zoom factor. By using this transformation, the focused spot can be zoomed in to show the details. But x-axis and y-axis must be zoomed with the same ratio, which is inconvenient in some cases. For example, slab lasers nowadays are widely used in industrial, military and medical areas. It generates powerful strip-shaped beams as shown in Fig. 9. At the focusing spot, the strip will be transposed due to lensing. If we keep the original strip-shaped window unchanged, we will not get the whole strip after lensing. Also we can not zoom in x-axis and zoom out y-axis at the same time. But if we decompose this process into low dimensional space, we find that the x and y direction can be operated independently, and we will have separate (α_x, z'_x) along x-axis and (α_y, z'_y) along y-axis. Consequently, we can zoom x-direction and y-direction freely.

The strip-shaped beam ($5mm \times 0.5mm$) is sampled by a 512×64 matrix, which is shown in Fig. 9. This beam is decomposed to 30 ROBs, and the singular values are shown in Fig. 10. The number of main singular values are no more than 23. By making these low dimensional beams pass through lens (focal length 100 mm) and space (length 100 mm) and at the same time using coordinate transformation method [8], we get the focused beam shown in Fig. 11. The focused beam is zoomed to a $0.8mm \times 5mm$ area with clear details. We calculate this process through Kirchhoff integral point by point (The calculating takes more than 30 minutes using a personal computer). Figure 12 shows the difference between the two methods, which is no more than 2%. We think it is acceptable, taking truncation error of SVD and Fresnel approximation error into consideration.

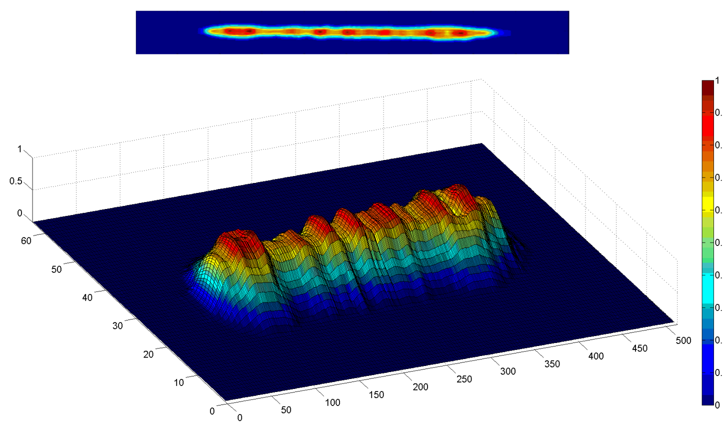

Fig. 9. Strip shaped beam generated from a slab laser. Upper: the beam in actual proportion. Lower: beam zoomed in.

6. Discussion

Experimental realization of low dimensional beams is possible. Firstly, it needs addition and multiplication operation of beams. Addition operation can be taken by using combining mirrors. Interference of two beams will realize the plural addition of the two beams. Multiplication is an important operation to realize transform from low dimensional beams to ROBs, which can be taken by employing second harmonic generation (SHG). SHG is a proper candidate, for it keeps phase information of the low dimensional beams, but changes the wavelength of composed beam. For example, we can use optical superlattice or surface plasmonic structure to generate two low dimensional beams as the x component and y component and expand low dimensional

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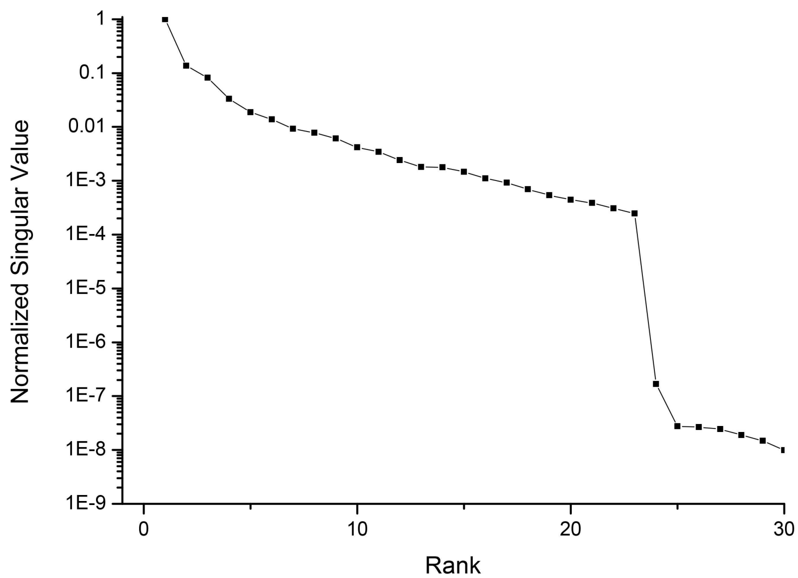

Fig. 10. Singular values of strip shaped beam.

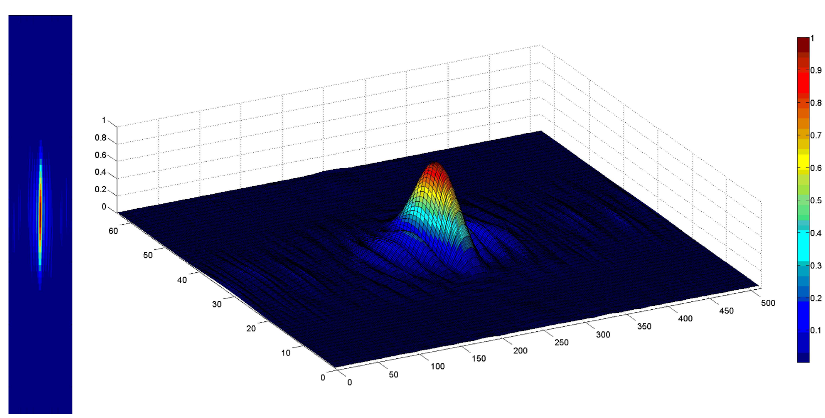

Fig. 11. Beam after lensing and free space propagation. Left: the beam in actual proportion. Right: beam zoomed in.

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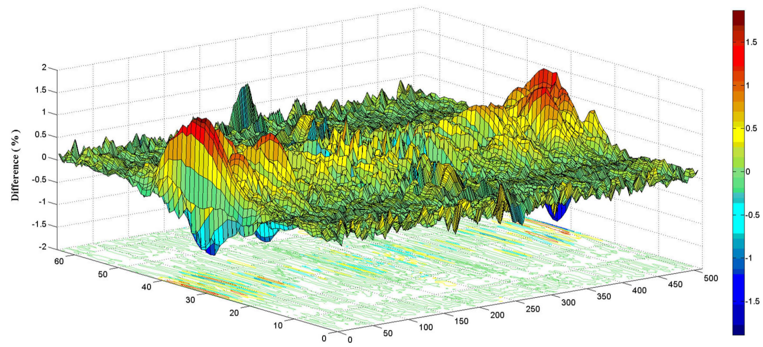

Fig. 12. Difference between low dimensional focusing and Kirchhoff integral.

beam to 2D beam by using cylindrical lens, then use SHG to multiply the two components to generate ROBs, and finally use combining mirrors to compose these ROBs. Decomposition of a beam into ROBs seems quite difficult. A tentative suggestion is to use cavity combining with SHG and optical parametric amplification (OPA), which will be discussed in the future.

7. Conclusion

In conclusion, we decompose optical beams and operators into low dimensional space via SVD. Singular values and ROBs (or ROOs) are the intrinsic characteristics of a given beam (or operator), regardless of the discretization. Complicated optical beams, e.g. vortex Bessel-Gussian beam, can be composed by limited number of low dimensional beams, suggesting a new approach to generate unconventional optical beams. Operation of beam is analyzed in this point of view and lensing and free space propagation is given as an example. In low dimensional space, orthogonal dimensions can be operated independently, which is illustrated in the focusing of a strip-shaped beam. Composition and decomposition may be realized experimentally by using nonlinear frequency conversion processes, e.g. SHG or OPA.

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